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# A Model of Oligopoly based on a Network Approach

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# Contents

# 1 Introduction

This thesis studies the competition of firms in one product market with network effect under which costs are dependent upon collaborations between firms. The idea of research is taken from [14], and concerns the question: what are the incentives of firms in market competition? The mentioned book covers many cases of market competitions and provides solution techniques. It discovers the topics of monopoly market and optimal behavior in quantities and prices, competition of many firms in one and many products market, price discrimination, dynamic competitions and so on. We focus on Cournot competition in quantities, the approaches of the research and development adoption of new technologies and cooperative game theory. In the book is proposed different solution concepts and models which concern these issues. The paper [2] provides an example of applying a model with research and development collaborations for non-cooperative and cooperative two-person games. Authors consider a two-stage game where on different stages actions of players represent the value of technological partnership and find Nash equilibria of the game. Another way of looking at the firms competition is to look precisely at their collaborations. A collaboration link can be interpreted as a partnership which is costly but lower costs of production of the firms involved. There can be many incentives for collaboration. Indeed technological partnerships, reduction of transportation and holding costs and others. The collaborations between firms can be represented by a network with firms settled in the nodes. In [9] M. Jackson describes social and economic networks, constructs models of behavior and analyses them using game theory and optimization methods. He provides allocation rules for cooperative games on networks as well. Mostly he focuses on the topology structure of equilibrium and stable networks. He discovers network formation stage, provides the conditions for existence of stable networks. He introduces a one-stage model of the game we discover, the game on a fixed network, but he does not inspect the two-stage game and consequently the issues of the two-stage equilibria and cooperative game. We use definitions, concepts and notations from this book in our work. In [8] authors discover a coordination game with the endogenous network structure with and costs of maintaining the collaborations. They examine stochastic stability issue on fixed networks, characterize stochastically stable states and inspect how the endogenous networks affect stochastic stability. Similar to ours, a non-cooperative model of network formation with link formation costs is investigated in [3]. There is considered one-way and two-way flow of benefits. The strict Nash equilibria are found in both models: for one-way flow model there are empty network and wheel network and for two-way flow model – empty and star networks. Also there is considered dynamic process and is proved that it converges to strict Nash equilibrium. Another close research is done in [6]. There Cournot oligopoly is considered with addition of opportunity for each firm to form pair-wise collaborative links with other firms which will lower costs of production of participants. The result is in the characterization of stable networks and comparison them with efficient networks. There is found that the complete network is stable. Authors also show that from a social point of view the complete network is efficient. The comprehensive overview of cooperative games and coalitional formations for applications in economics is provided in [4]. There are discovered general issues of incentives to cooperate, form a coalition, provided analysis of influence groups of coalitions to other coalitions, examined the bargaining issue of total payoff of coalition between players. And there is considered competition of coalitions. In [13] there is an analysis of cooperative game based on network model with costs for establishing links using an extension of the Myerson value to determine the payoffs in a 3-player symmetric game and the issue of existence coalition proof Nash equilibria in the 3-player symmetric game. In [7] authors develop a model of oligopoly market with the network effect on payoff functions and examine the incentives of firms to form collaborations with other firms. They find the nature of collaboration structures that are stable under different market conditions, and characterized stable structures. Unlike stability issue in [7] here we inspect Nash equilibria. We decide to discover the firms competition from the two points of view at the same time: quantities competition and network formation. As the basis of such analysis we use [11]. The paper provides analysis of links' influence on strategy choice of a player for a general payoff function. The issue of dynamic stability of cooperation solutions is examined.

The dissertation is based on these works. As in [11] we consider a two-stage game of $n$ firms where at the first stage players form the network of collaborations and at the second stage the firms chose quantities of production as in [7]. After these two stages payoffs are computed and the game ends. This game illustrates the competition of firms in one-product market. Our first aim is to find equilibria, characterize them by profitability and network topology structure. We establish preferred equilibria and provide sensitivity analysis of the player's behavior and the market performance. The second goal is to find the cooperative solution of the game and compare it with non-cooperative solution. We examine a two-stage oligopoly model from [7] with offering costs as well. It differs from the previous model in the payoff function in such a way that an incentive to form a collaboration link induces additional costs. In this model we find sufficient condition for equilibria. We should notice that in the papers above the issue of equilibria in two-stages games is not discussed as like as the cooperative solution of firms competition in two-stages and our work tries to figure out firms equilibrium behavior and common laws which helps to better understand how firms should act in one-product market competition: should they play as singletones or cooperate, how the collaborations influence on different players and what concrete actions they should do to benefit.

The paper has the following structure. In section 2 we investigate the non-cooperative two-stage oligopoly. At first we define the model, strategies and payoff functions. Then we find an equilibrium when the network is fixed. After this we construct a hypothesis of equilibrium network topology structure and test it. Next we answer the question which equilibria are more profitable for players and how it concerns other players. In sensitivity analysis we explore how the adding or removing the link affect player's equilibrium strategies, payoffs and price function. The special case of regular network is explored in detail and with an example. At the end of this section we adopt the cost function for the weighted networks and say how the equilibrium action for fixed network will change. Section 3 is a consideration of cooperative game approach. We investigate both models: with full cooperation on two stages and cooperation only on quantities competition stage. We give an overview of methods of construction characteristic function, and introduce solution concepts of the bargaining total payoff which we will use. The characteristic function then is chosen as the value maximin optimization problem. The Shapley value [12] and the center-of-gravity of the imputation set (CIS value) [5] are used as imputations. Finally the sensitivity analysis of cooperative game is provided. Section 4 introduces the model of two-stage oligopoly with offering costs which we examined. The methods of analysis the last model coincide with the previous two-stage oligopoly model.

# 2 Two-stage oligopoly

## 2.1 The model

We consider a two-stage game of $n$ players. At first stage players offer collaboration links to each other simultaneously. After all players made their offers pairwise links form a network, one way links are removed. At the end of the first stage we have a formed undirected network connecting players. At the second stage there is Cournot competition in quantities. Players choose quantities of production simultaneously. The second stage ends when all players have made their choices. After second stage the payoffs are calculated. Payoff functions are dependent upon quantities and they also dependent upon network. After this step the game ends. We now develop the required terminology and provide some definitions.

### 2.1.1 Strategies

Let $N = \{1, ..., n\}$ be a finite set of players. A pair $(N, g)$, where $N$ is a set of players and $g$ setting the topology of collaborations between players, we call a network.

At the first stage, network formation appears. Players simultaneously choose their actions – $n$-dimensional vectors $g_i = (g_{i1}, \ldots, g_{in}),\ i \in N$ with components defined as:

$$g_{ij} = \begin{cases} 1, & \text{if player } i \text{ offers a link to player } j \in M, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

If an element $(i, j) \in g$, it means that there exists a link between player $i$ and player $j$. To simplify notations, we will identify the network $g$ with the action profile $(g_1, \ldots, g_n)$ and denote action profile by $g$. A link $(i, j)$ will be denoted by $ij$. Let $g_{-i} = (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$ denotes an action profile $g$ without player $i$'s action. Using our notations the equality $g = (g_i, g_{-i})$ holds. Number of neighbors of player $i$ in the network $g$ is the degree of node $i$ in the network $g$ and denoted by $\eta_i(g) = |\{j \in N \setminus \{i\} : ij \in g\}|$.

Once the action profile $(g_1, g_2, \dots, g_n)$ has chosen, it defines a network $g$ in the following way: a link $ij$ is formed and consequently belongs to network $g$ only if $g_{ij} = g_{ji} = 1$, i.e., both players agree to form it. At the end of the network formation stage, network $g$ is realized.

At the second stage players compete in choosing the quantities, i.e. Cournot competition is played. The action of player $i$ at the second stage is quantity $q_i \in [0, \overline{q}]$, where $\overline{q}$ is sufficiently large. Players choose their actions at the same time. At the end of this stage the action profile $q = (q_1, \dots, q_n)$ is formed.

After two stages player $i$ has two actions – action $g_i$ from the first stage and action $q_i$ from the second stage. These actions form the strategy $(g_i, q_i)$ player $i$ in the two-stage game. All strategies of all players form the strategy profile $((g_1, q_1), \dots, (g_n, q_n))$ in the game.

#### 2.1.2 Payoff function

We come out from the assumption that collaborations lower marginal costs of production. A network $g$, therefore, induces a marginal costs for the firms which is given by $c_1(g), c_2(g), \dots, c_n(g)$. We assume that firm $i$'s marginal cost in the network $g$ is a function of the number of collaboration links it has with other firms and is strictly decreasing in the number of these links:

$$c_i(g) = c(\eta_i(g)),\ c(\eta_i(g) + 1) < c(\eta_i(g)),\ i \in N. \tag{2}$$

To rule out uninteresting cases, we will assume that $c_i(g) \geqslant 0$, $\forall i \in$

$N$, $\forall g$. We assume that marginal costs are linearly declining in the number of links, i.e.

$$c_i(g) = \gamma_0 - \gamma\eta_i(g), \; i \in N, \tag{3}$$

where $\gamma_0 > 0$, represents a firm's marginal cost when it has no links, while $\gamma > 0$ is the cost reduction induced by each link formed by a firm. The assumption of non-negativeness of the marginal costs leads us to the following constraint:

$$\gamma_0 \geqslant \gamma(n-1). \tag{4}$$

Suggest the following linear inverse market demand function:

$$p(q) = \alpha - \sum_{i \in N} q_i, \; \alpha > 0. \tag{5}$$

We suppose that $\alpha$ is sufficiently large.

And finally define the payoff function on a network $g$ for player $i \in N$ as follows:

$$\pi_i(g, q) = (p - c_i(g))q_i. \tag{6}$$

## 2.2 Equilibrium at fixed network

At the second stage we have fixed undirected network $g$. At this stage player's action is its quantity $q_i$. In other words we have Cournot competition in quantities.

The necessary first-order condition for action profile $q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ to be a Nash equilibrium is that for each firm $i \in N$

$$\left.\frac{\partial \pi_i(g, q)}{\partial q_i}\right|_{q^*} = 0. \tag{7}$$

The sufficient second-order condition for action profile $q^* = (q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium is that $q_i^*$ yields a maximum of $\pi_i(g, q)$, put differently, strict concavity of payoff function $\pi_i(g)$, i.e.

$$\left.\frac{\partial^2 \pi_i(g,q)}{\partial q_i^2}\right|_{q^*} < 0, \ \forall i \in N. \tag{8}$$

Let us first check the sufficient second-order condition.

$$\frac{\partial^2 \pi_i(g,q)}{\partial q_i^2} = \frac{\partial \left( q_i \frac{\partial p}{\partial q_i} + (p - c_i(g)) \right)}{\partial q_i} = q_i \frac{\partial^2 p}{\partial q_i^2} + 2\frac{\partial p}{\partial q_i} = -2 < 0 \tag{9}$$

We have demonstrated that each firm's profit is strictly concave for any given action profile $(q_1, q_2, \ldots, q_n)$ and any network $g$. Therefore the second-order condition is satisfied and, furthermore, the first-order condition is sufficient for action profile $(q_1^*, q_2^*, \ldots, q_n^*)$ to be a Nash equilibrium.

Let us find the Nash equilibrium from the necessary first-order condition for a Nash equilibrium. We have the system of payoff functions for all players:

$$\begin{cases} \pi_1(g,q) = (p(q) - c_1(g))q_1, \\ \pi_2(g,q) = (p(q) - c_2(g))q_2, \\ \ldots \\ \pi_n(g,q) = (p(g) - c_n(g))q_n. \end{cases} \tag{10}$$

Below the process of finding the equilibrium output is shown.

$$\begin{cases} \frac{\partial \pi_1(g,q)}{\partial q_1} = q_1 \frac{\partial p}{\partial q_1} + p - c_1(g) = 0, \\ \frac{\partial \pi_2(g,q)}{\partial q_2} = q_2 \frac{\partial p}{\partial q_2} + p - c_2(g) = 0, \\ \ldots \\ \frac{\partial \pi_n(g,q)}{\partial q_n} = q_n \frac{\partial p}{\partial q_n} + p - c_n(g) = 0. \end{cases} \tag{11}$$

When we substitute $p(q)$ from (5) and $c_i(g)$ from (3) into $i$-th equation we obtain:

$$\begin{cases} q_1(-1) + \alpha - \sum_{i \in N} q_i - \gamma_0 + \gamma\eta_1(g) = 0, \\ q_2(-1) + \alpha - \sum_{i \in N} q_i - \gamma_0 + \gamma\eta_2(g) = 0, \\ \dots \\ q_n(-1) + \alpha - \sum_{i \in N} q_i - \gamma_0 + \gamma\eta_n(g) = 0. \end{cases} \tag{12}$$

Sum up all equations:

$$-\sum_{i \in N} q_i + n(\alpha - \gamma_0) - n\sum_{i \in N} q_i + \gamma\sum_{i \in N} \eta_i(g) = 0$$

$$n(\alpha - \gamma_0) + \gamma\sum_{i \in N} \eta_i(g) = (n+1)\sum_{i \in N} q_i$$

$$\sum_{i \in N} q_i = \frac{n(\alpha - \gamma_0) + \gamma\sum_{i \in N} \eta_i(g)}{n+1} \tag{13}$$

Look at the $i$-th equation of system (11):

$$q_i + \sum_{i \in N} q_i = \alpha - \gamma_0 + \gamma\eta_i(g) \tag{14}$$

After substitution (13) into (14), we get the following

$$q_i + \frac{n(\alpha - \gamma_0) + \gamma\sum_{i \in N} \eta_i(g)}{n+1} = \alpha - \gamma_0 + \gamma\eta_i(g) \tag{15}$$

$$q_i = \alpha - \gamma_0 + \gamma\eta_i(g) - \frac{n(\alpha - \gamma_0) + \gamma\sum_{i \in N} \eta_i(g)}{n+1} \tag{16}$$

Finally Cournot equilibrium quantities can be written as follows

$$q_i^*(g) = \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j \neq i} \eta_j(g)}{n+1}, \ i \in N. \tag{17}$$

At this point we found the optimal quantity of production for every player. The equilibrium profit (6) for player $i$ is

$$\pi_i(g, q^*) = (p(q^*) - c_i(g))q_i^*(g) = \Big(\alpha - \sum_{i\in N} q_i^*(g) - \gamma_0 + \gamma\eta_i(g)\Big)q_i^*(g). \quad (18)$$

Comparing the expression in brackets $\alpha - \sum_{i\in N} q_i^* - \gamma_0 + \gamma\eta_i(g)$ with the formula (14) we come to the final result. For a given network $g$, Cournot profit for firm $i \in N$ has the following form

$$\pi_i(g) = q_i^{*2}(g). \quad (19)$$

**Proposition 1.** *For a fixed network there is a unique equilibrium in competition in quantities. The optimal quantity for firm i is*

$$q_i^*(g) = \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(g)}{n+1}. \quad (20)$$

*The payoff function for i-th firm has the form of*

$$\pi_i(g) = q_i^{*2}(g). \quad (21)$$

In order to ensure that each firm produces a strictly positive quantity in equilibrium, consider the worst case for $i$-th firm – when firm $i$ has no any links in formed network, and all the remaining firms $N \setminus \{i\}$ form a complete network. The quantity of firm $i$ has to be positive

$$q_i^*(g) = \frac{\alpha - \gamma_0 - (n-1)(n-2)\gamma}{n+1} > 0. \quad (22)$$

Finally by simplifying the last inequality we obtain

$$(\alpha - \gamma_0) - (n-1)(n-2)\gamma > 0. \quad (23)$$

The inequality (23) actually gives a lower bound for $\alpha$.

## 2.3 Equilibrium in the two-stage game

In the two-stage game the strategy of player $i$ is a pair $(g_i, q_i)$, where $g_i$ is his pure action at the network formation stage which represents his desirable collaborations with other players and $q_i$ is his pure action at the quantity competition stage when collaborations of all players are already fixed after network formation stage. Consequently we have strategy profile $((g_1, q_1), \ldots, (g_n, q_n))$, where $g$ is a network which is obtained after all players chose desirable links $g_i$, and quantities $q_i$, $i \in N$. In a simple form strategy profile can be written as pair $(g, q)$.

The goal of this section is to find Nash equilibria in the two-stage game. Of course the problem of finding all Nash equilibria in the infinite set of strategy profiles is very complex so we will make a hypothesis of structure Nash equilibria in specific networks and check it.

Assume that network $g$ is a pairwise network, i.e. for any offered link complementary link is offered as well. In network notations it means $g_{ij} = g_{ji}$. An illustration of such networks with 3 players is shown on Figure 1.
Check whether such network is a Nash equilibrium and which constraints we should apply to say that such network $g$ is a Nash equilibrium.

When in a pairwise network player $i$ deviates from his action $g_i^*$ with fixed actions $g_j^*$ , $j \neq i$, he cannot increase the number of his collaborations. The number of neighbors of the player may stay the same or may be less than in pairwise network, because other players do not deviate and consequently do not propose new links. An example of the deviation is shown on Figure 2. There picture a) demonstrates pairwise network $g^*$ and pictures b) and c) show deviations of player 1. Formally saying, given a pairwise network $g^*$, deviation of player $i$ from the action $g_i^*$ to the action $g_i$ can be expressed in

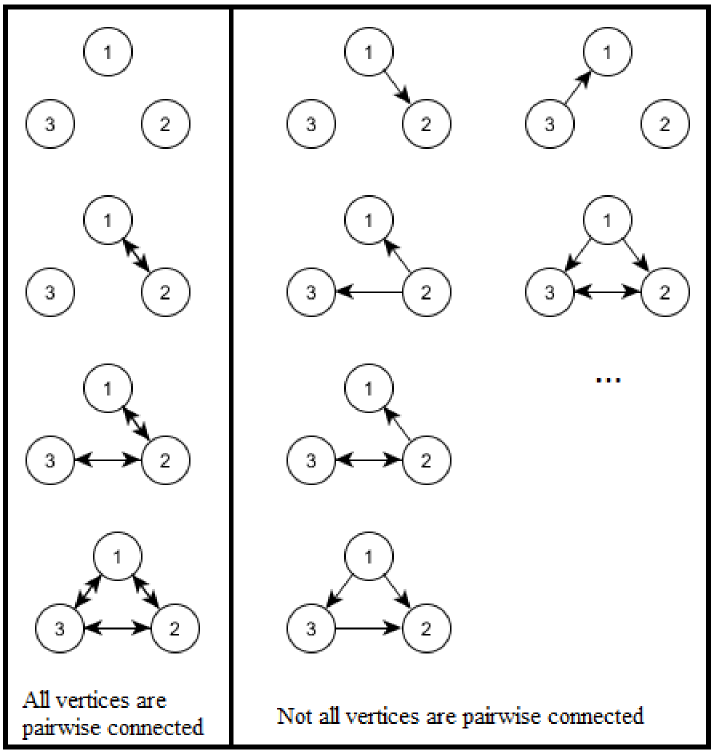


Figure 1: Pairwise and not pairwise networks.

the following form:

$$\eta_i(g^*) - \eta_i(g^*||g_i) = l,\ l = 0, 1, \ldots, \eta_i(g^*). \tag{24}$$

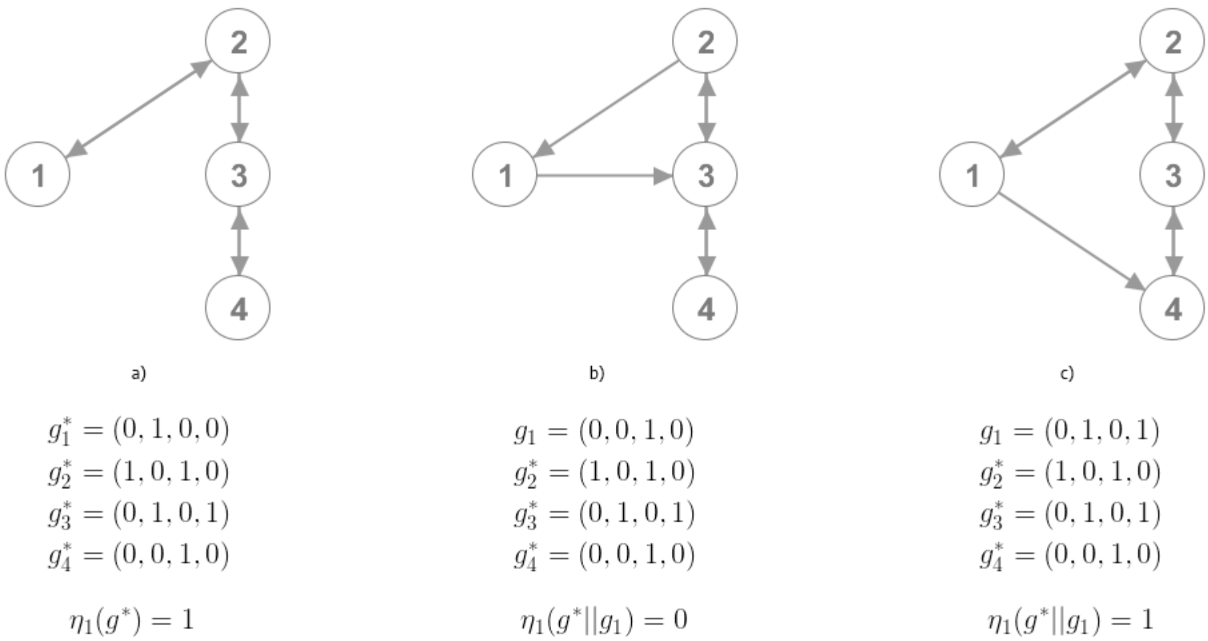


Figure 2: Example of deviations from $g^*$ at network formation stage.

By definition, the strategy profile $(g^*, q^*)$ is a Nash equilibrium if player $i$ does not get a surplus from his deviation with fixed strategies of other players. Formally, $(g^*, q^*)$ is a Nash equilibrium if

$$\pi_i(g^*, q^*) \geqslant \pi_i(g^*, q^*||g_i, q_i),\ \forall g_i, \forall q_i, \forall i \in N \tag{25}$$

Fix such $q_i$ which maximizes $\pi_i(g^*, q^*||g_i, q_i)$. It will be sufficient for holding the inequality above. Indeed if the inequality above holds for all $q_i$ then it holds for such specific $q_i$ which maximizes the right-hand side of (25), i.e.,

$$\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) \geqslant \pi_i(g^*, q^*||g_i, q_i),\ \forall g_i, \forall q_i, \forall i \in N. \tag{26}$$

By the same logic we can fix such $g_i$ that maximizes $\max_{q_i} \pi_i(g^*, q^*||g_i, q_i)$ and consequently get rid off variability in strategies in network formation stage.

$$\max_{g_i} \max_{q_i} \pi_i(g^*, q^*||g_i, q_i) \geqslant \max_{q_i} \pi_i(g^*, q^*||g_i, q_i),\ \forall g_i, \forall i \in N \tag{27}$$

After combining inequalities (26), (27) and having the reasoning above the inequality (25) proceeds to the following inequality:

$$\pi_i(g^*, q^*) \geqslant \max_{g_i} \max_{q_i} \pi_i(g^*, q^*||g_i, q_i),\ \forall i \in N \tag{28}$$

At first we need to solve the maximization problem over quantity from the right-hand side of (25):

$$\max\nolimits_{q_i} \pi_i(g^*, q^*||g_i, q_i) = \tag{29}$$

$$= \max\nolimits_{q_i} \Big(\alpha - \textstyle\sum_{j \neq i} q_j^*(g^*) - q_i - \gamma_0 + \gamma \eta_i(g^*||g_i)\Big)\, q_i \tag{30}$$

Actually it is a common maximization problem of one variable and to find the solution it is needed to take the derivative of $\pi_i(g^*, q^*||g_i, q_i)$ with respect to $q_i$:

$$\frac{\partial}{\partial q_i}\pi_i(g^*, q^*||g_i, q_i) = 0 \tag{31}$$

$$\frac{\partial}{\partial q_i}\left[\left(\alpha - \sum_{j\neq i} q_j^*(g^*) - q_i - \gamma_0 + \gamma\eta_i(g^*||g_i)\right) q_i\right] = 0 \tag{32}$$

$$\alpha - \sum_{j\neq i} q_j^*(g^*) - 2q_i - \gamma_0 + \gamma\eta_i(g^*||g_i) = 0 \tag{33}$$

Finally we obtain such $q_i$ that maximizes $\pi_i(g^*, q^*||g_i, q_i)$:

$$q_i = \frac{1}{2}\left(\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)\right) \tag{34}$$

Substitute $q_i$ from the formula above into the $\pi_i(g^*, q^*||g_i, q_i)$ to obtain $\max_{q_i} \pi_i(g^*, q^*||g_i, q_i)$:

$$\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) =$$

$$= \left[\alpha - \sum_{j\neq i} q_j^*(g^*) - q_i - \gamma_0 + \gamma\eta_i(g^*||g_i)\right] q_i =$$

$$= \left[\alpha - \sum_{j\neq i} q_j^*(g^*) - \frac{\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)}{2} - \gamma_0 + \gamma\eta_i(g^*||g_i)\right] \times$$

$$\times \frac{\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)}{2} =$$

$$= \left(\frac{\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)}{2}\right)^2 =$$

$$= \frac{1}{4}\left(\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)\right)^2 =$$

$$= \frac{1}{4}\left(\alpha - \sum_{j\in N} q_j^*(g^*) + q_i^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)\right)^2 =$$

Substitute formula (13) instead of sum $\sum_{j\in N} q_j^*(g^*)$ and formula (20) instead of $q_i^*(g^*)$ into the last expression above:

$$= \frac{1}{4}\Big(\alpha - \frac{n(\alpha-\gamma_0) + \gamma\sum_{i\in N}\eta_i(g^*)}{n+1} +$$
$$+ \frac{\alpha - \gamma_0 + n\gamma\eta_i(g^*) - \gamma\sum_{j\neq i}\eta_j(g^*)}{n+1} - \gamma_0 + \gamma\eta_i(g^*||g_i)\Big)^2 =$$
$$= \left(\frac{\alpha - \gamma_0 - \gamma\sum_{i\in N}\eta_i(g^*)}{n+1} + \frac{\gamma}{2}\Big(\eta_i(g^*) + \eta_i(g^*||g_i)\Big)\right)^2 =$$
$$= \left(q_i^*(g^*) + \frac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2$$

We obtain the maximum over quantity (when player $i$ is deviating in quantity) of the right-hand side of the (25):

$$\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) = \left(q_i^*(g^*) + \frac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 \tag{35}$$

Now we need to solve the maximization over $g_i$ problem:

$$\max_{g_i}\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) = \max_{g_i}\left(q_i^*(g^*) + \tfrac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 \tag{36}$$

The last expression can be transformed to the following:

$$\max_{g_i}\left(q_i^*(g^*) + \frac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 = \tag{37}$$

$$= \left(q_i^*(g^*) + \frac{\gamma}{2}\max_{g_i}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 \tag{38}$$

According to the (24) we can conclude that:

$$\max_{g_i} \Big( \eta_i(g^*||g_i) - \eta_i(g^*) \Big) = 0 \tag{39}$$

Finally we obtain the right-hand side (when player $i$ is deviating at both stages) of Nash equilibrium definition (25):

$$\max_{g_i} \max_{q_i} \pi_i(g^*, q^*||g_i, q_i) = \left(q_i^*(g^*)\right)^2 \tag{40}$$

Since

$$\pi_i(g^*, q^*) = \left(q_i^*(g^*)\right)^2, \; i \in N, \tag{41}$$

now we can be sure that the condition for Nash equilibrium (25) is always satisfied. Consequently we proved the following result.

**Proposition 2.** *All pairwise networks are the Nash equilibrium in the two-stage game.*

## 2.4 Preferred equilibria

From the form of the payoff function in equilibrium (19) and the sufficient condition for Nash equilibrium (Proposition 2) we can conclude that some equilibria may be more profitable for some players than others.

For example, the regular network is more profitable equilibrium than the empty network. Moreover the greater degree of node in regular network the greater payoff players get relatively to the payoff in regular network with less degree. Figure 3 illustrates that for 6 players 0-regular network is less profitable than 1-regular network, and that 1-regular network is less profitable than a complete network for any player.

Indeed, payoffs for the empty network and regular network are in the

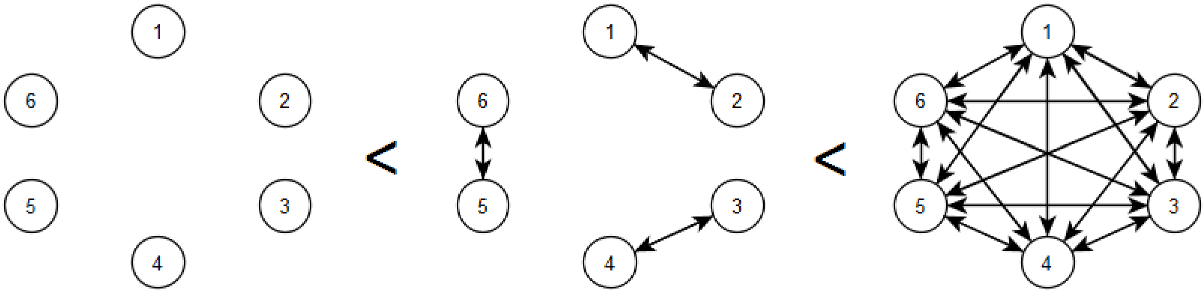


Figure 3: The greater degree of node in regular network the greater benefit players get.

following relation:

$$\pi_i(\text{empty}) = \left(\tfrac{\alpha-\gamma_0}{n+1}\right)^2 <$$

$$\leqslant \pi_i(k\text{-regular network}) = \left(\tfrac{\alpha-\gamma_0}{n+1} + \gamma\tfrac{k}{n+1}\right)^2 \tag{42}$$

$$\leqslant \pi_i(\text{complete}) = \left(\tfrac{\alpha-\gamma_0}{n+1} + \gamma\left(1 - \tfrac{1}{n+1}\right)\right)^2 \;\; \forall i \in N \tag{43}$$

Another interesting situation – star network. We call a player in the star network the central player if he has the highest degree in the star network. We observed that the star network central player benefits, others do not. Figure 4 shows that the star network is more profitable for central player.

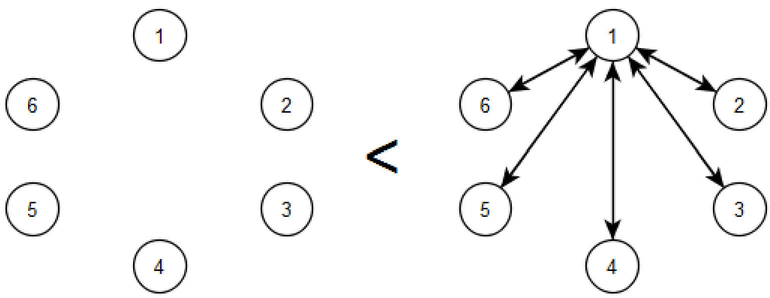


Figure 4: Only player 1 benefits, others lose.

In this case the following inequality for players' payoffs hold: for the player, located in the central node, we have:

$$\pi_{central}(\text{empty}) = \left(\tfrac{\alpha-\gamma_0}{n+1}\right)^2 <$$

$$< \pi_{central}(\text{star network}) = \left(\tfrac{\alpha-\gamma_0}{n+1} + \gamma\tfrac{k}{n+1}\right)^2, \tag{44}$$

and for any other player $k$ holds:

$$\pi_k(\text{empty}) = \left(\frac{\alpha-\gamma_0}{n+1}\right)^2 <$$
$$< \pi_k(\text{star network}) = \left(\frac{\alpha-\gamma_0}{n+1} - \gamma\frac{n-3}{n+1}\right)^2 \ \forall n \geqslant 3 \tag{45}$$

Hence we come up to the following propositions:

**Proposition 3.** *A $k$-regular network is more profitable than an $l$-regular network for any player $i \in N$ if $l < k$.*

**Proposition 4.** *A star network is more profitable than empty network only for the player located in the central node.*

## 2.5 Sensitivity analysis

We will use equilibrium quantities (17) for given network $g$ which are dependent on a network structure so for the simplification we will discard the parameter $q$ in the payoff function (6).

Suppose that player $i$ in given network $g$ deletes the link with player $j$. After this transformation network $g$ changes and we denote the new network by $\tilde{g}$. An example of such situation is illustrated on Figure 5.

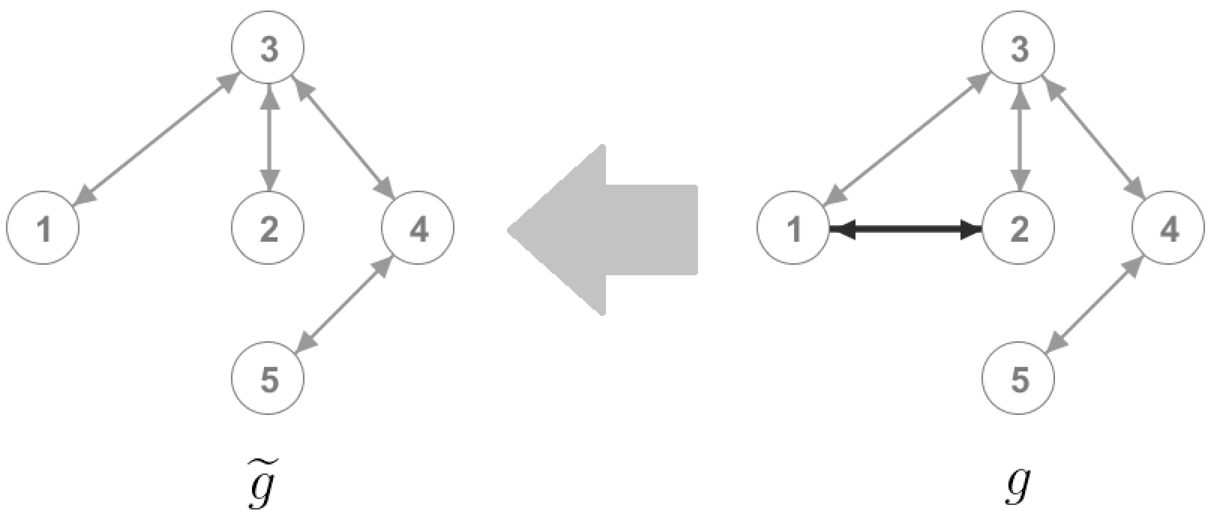


Figure 5: Example of removing the link in the network.

The equilibrium quantity for player $i$ in the new network $\tilde{g}$ has changed in the following way:

$$\begin{aligned}
q_i^*(\tilde{g}) &= \frac{\alpha - \gamma_0 + n\gamma\big(\eta_i(g) - 1\big) - \gamma\big(\sum_{j\neq i}\eta_j(g) - 1\big)}{n+1} \\
&= \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(g)}{n+1} - \gamma\frac{n-1}{n+1} \\
&= \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(g)}{n+1} - \gamma\left(1 - \frac{2}{n+1}\right) \\
&= q_i^*(g) - \gamma\left(1 - \frac{2}{n+1}\right) \qquad (46)
\end{aligned}$$

The quantity $q_j^*(\tilde{g})$ for player $j$ is changed by the same rule. Now let us consider how removing of the link $(ij)$ affects the equilibrium quantity of other player $k \neq i, j$.

$$\begin{aligned}
q_k^*(\tilde{g}) &= \frac{\alpha - \gamma_0 + n\gamma\eta_k(g) - \gamma\big(\sum_{j\neq k}\eta_j(g) - 2\big)}{n+1} \\
&= \frac{\alpha - \gamma_0 + n\gamma\eta_k(g) - \gamma\sum_{j\neq k}\eta_j(g)}{n+1} + \gamma\frac{2}{n+1} \\
&= q_k^*(g) + \gamma\frac{2}{n+1} \qquad (47)
\end{aligned}$$

Consider influence of the removing link in the network to the price function (5):

$$\begin{aligned}
p(\tilde{g}) &= \alpha - \sum_{i\in N} q_i^*(\tilde{g}) = \\
&= \alpha - \sum_{i\in N} q_i^*(g) + 2\gamma\left(1 - \frac{2}{n+1}\right) - (n-2)\gamma\frac{2}{n+1} = \\
&= \alpha - \sum_{i\in N} q_i^*(g) + \gamma\left(2\left(1 - \frac{2}{n+1}\right) - (n-2)\frac{2}{n+1}\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \alpha - \sum_{i \in N} q_i^*(g) + \gamma \left(2 - \frac{4}{n+1} + \frac{4-2n}{n+1}\right) = \\
&= \alpha - \sum_{i \in N} q_i^*(g) + 2\gamma \left(1 - \frac{n}{n+1}\right) = \\
&= \alpha - \sum_{i \in N} q_i^*(g) + \gamma \frac{2}{n+1} = \\
&= p(g) + \gamma \frac{2}{n+1} \qquad (48)
\end{aligned}
$$

We observe the positive correlation of price with the removing of link.

Consider now players' payoffs. At first look at the payoff function of player $i$ (19) in the new network $\tilde{g}$.

$$
\begin{aligned}
\pi_i(\tilde{g}) &= \left(q_i^*(\tilde{g})\right)^2 = \\
&= \left(q_i^*(g) - \gamma\left(1 - \frac{2}{n+1}\right)\right)^2 = \\
&= \left(q_i^*(g)\right)^2 - 2q_i^*(g)\gamma\left(1 - \frac{2}{n+1}\right) + \left(\gamma\left(1 - \frac{2}{n+1}\right)\right)^2 = \\
&= \pi_i(g) + \gamma\left(1 - \frac{2}{n+1}\right)\left(\gamma\left(1 - \frac{2}{n+1}\right) - 2q_i^*(g)\right)
\end{aligned}
$$

Let us look under which constraint player $i$ (and player $j$) benefits from the removing the link $(ij)$. Due to the form of the payoff function of player $i$ in the network $\tilde{g}$ and non-negativeness of term $\gamma\left(1 - \frac{2}{n+1}\right)$, the payoff of player $i$ has a positive correlation with the deletion of the collaboration if the next condition holds:

$$
\gamma\left(1 - \frac{2}{n+1}\right) - 2q_i^*(g) \geqslant 0, \qquad (49)
$$

or

$$
q_i^*(g) \leqslant \frac{1}{2}\gamma\left(1 - \frac{2}{n+1}\right). \qquad (50)
$$

The payoff of player $k \neq i, j$ in the network $\tilde{g}$ has the following form:

$$\begin{aligned} \pi_k(\tilde{g}) &= (q_k^*(\tilde{g}))^2 = \\ &= \left(q_k^*(g) + \gamma\frac{2}{n+1}\right)^2 = \\ &= (q_k^*(g))^2 + 2q_k^*(g)\gamma\frac{2}{n+1} + \left(\gamma\frac{2}{n+1}\right)^2 = \\ &= \pi_k(g) + \gamma\frac{4}{n+1}\left(q_k^*(g) + \gamma\frac{1}{n+1}\right) \geqslant \pi_k(g). \end{aligned} \tag{51}$$

We can say that payoff for player $k \neq i, j$ does not decrease after players $i$ and $j$ formed a new connection. In general, if the condition (50) holds, players $i, j$ profit in the network $\tilde{g}$ in comparison to the network $g$. In contrast player $k$ always gains in the network $\tilde{g}$ in comparison to the network $g$.

Now suppose that player $i$ in given network $g$ suggests a link to some other player $j$ and the last one accepts it. It means that one new link is added to the network $g$. Denote this new network by $\tilde{g}$. An example of such situation is illustrated on Figure 6.

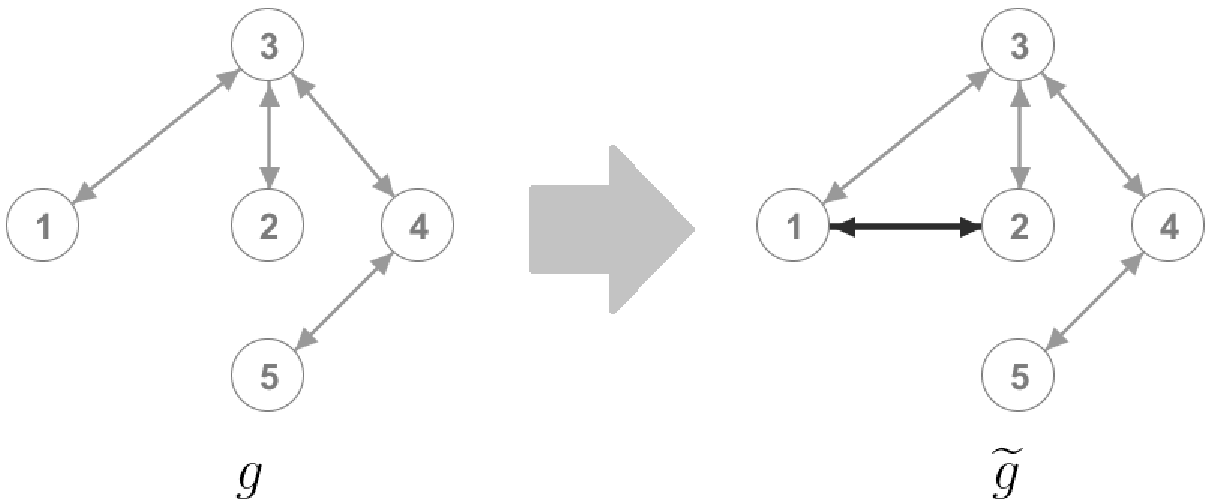


Figure 6: Example of establishing a new link in the network.

The equilibrium quantity for player $i$ in this new network $\tilde{g}$ has changed in the following way:

$$
\begin{aligned}
q_i^*(\tilde{g}) &= \frac{\alpha - \gamma_0 + n\gamma\big(\eta_i(g)+1\big) - \gamma\big(\sum_{j\neq i}\eta_j(g)+1\big)}{n+1} \\
&= \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(g)}{n+1} + \gamma\frac{n-1}{n+1} \\
&= \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(g)}{n+1} + \gamma\left(1 - \frac{2}{n+1}\right) \\
&= q_i^*(g) + \gamma\left(1 - \frac{2}{n+1}\right). \qquad (52)
\end{aligned}
$$

The quantity $q_j^*(\tilde{g})$ for player $j$, who has accepted the link offered by player $i$, is changed by the same rule. We can make an important conclusion from the last equation: the number of collaborations of the player is positively correlated to the equilibrium quantity of production of the player while other players do not deviate.

Now let us consider how the addition of a new link affects equilibrium quantity of other player $k \neq i, j$ who is not involved in the new collaboration between players $i$ and $j$.

$$
\begin{aligned}
q_k^*(\tilde{g}) &= \frac{\alpha - \gamma_0 + n\gamma\eta_k(g) - \gamma\big(\sum_{j\neq k}\eta_j(g)+2\big)}{n+1} \\
&= \frac{\alpha - \gamma_0 + n\gamma\eta_k(g) - \gamma\sum_{j\neq k}\eta_j(g)}{n+1} - \gamma\frac{2}{n+1} \\
&= q_k^*(g) - \gamma\frac{2}{n+1}. \qquad (53)
\end{aligned}
$$

We can see that for the player $k \neq i, j$ the equilibrium quantity does not decrease with appearance of the link between players $i$ and $j$. And the amount of this reduce has a negative correlation with the number of players in the game: the more players in the game the less reduction of quantity player $k$ should do if players to stay in the Nash equilibrium.

Consideration of the price function uncovers a negative correlation between the quantity with the number of collaborations:

$$\begin{aligned}
p(\tilde{g}) &= \alpha - \sum_{i\in N} q_i^*(\tilde{g}) = \\
&= \alpha - \sum_{i\in N} q_i^*(g) - 2\gamma\left(1 - \frac{2}{n+1}\right) + (n-2)\gamma\frac{2}{n+1} = \\
&= \alpha - \sum_{i\in N} q_i^*(g) - \gamma\left(2\left(1 - \frac{2}{n+1}\right) - (n-2)\frac{2}{n+1}\right) = \\
&= \alpha - \sum_{i\in N} q_i^*(g) - \gamma\left(2 - \frac{4}{n+1} + \frac{4-2n}{n+1}\right) = \\
&= \alpha - \sum_{i\in N} q_i^*(g) - 2\gamma\left(1 - \frac{n}{n+1}\right) = \\
&= \alpha - \sum_{i\in N} q_i^*(g) - \gamma\frac{2}{n+1} = \\
&= p(g) - \gamma\frac{2}{n+1} \qquad (54)
\end{aligned}$$

Consider now players' payoffs. At first look at the payoff function of player $i$ (19) in the new network $\tilde{g}$.

$$\begin{aligned}
\pi_i(\tilde{g}) &= (q_i^*(\tilde{g}))^2 = \\
&= \left(q_i^*(g) + \gamma\left(1 - \frac{2}{n+1}\right)\right)^2 = \\
&= (q_i^*(g))^2 + 2q_i^*(g)\gamma\left(1 - \frac{2}{n+1}\right) + \left(\gamma\left(1 - \frac{2}{n+1}\right)\right)^2 = \\
&= \pi_i(g) + \gamma\left(1 - \frac{2}{n+1}\right)\left(\gamma\left(1 - \frac{2}{n+1}\right) + 2q_i^*(g)\right) \geqslant \pi_i(g)
\end{aligned}$$

It means that addition of a new link $(ij)$ is always profitable for players $i$ and $j$.

The payoff of player $k \neq i, j$ in the network $\tilde{g}$ has the following form:

$$\begin{aligned}
\pi_k(\tilde{g}) &= (q_k^*(\tilde{g}))^2 = \\
&= \left(q_k^*(g) - \gamma\frac{2}{n+1}\right)^2 = \\
&= (q_k^*(g))^2 - 2q_k^*(g)\gamma\frac{2}{n+1} + \left(\gamma\frac{2}{n+1}\right)^2 = \\
&= \pi_k(g) + \gamma\frac{4}{n+1}\left(\gamma\frac{1}{n+1} - q_k^*(g)\right).
\end{aligned} \tag{55}$$

Hence, we obtain the following inequality:

$$\pi_k(\tilde{g}) \geqslant \pi_k(g) \tag{56}$$

if the following condition holds:

$$\gamma\frac{1}{n+1} - q_k^*(g) \geqslant 0, \tag{57}$$

or

$$q_k^*(g) \leqslant \gamma\frac{1}{n+1}. \tag{58}$$

We can say that payoff for player $k \neq i, j$ does not increase after players $i$ and $j$ formed a new connection if the condition (58) holds. In general while players $i, j$ always gain, player $k$ profits in network $\tilde{g}$ in comparison to the network $g$ if the condition (58) holds.

We may notice one more interesting property of the game: the more players are in the game the more quantity and payoff of player increase when the player establishes a new connection in network. And consequently the common price of product decreases.

### 2.5.1 Regular networks

Examine how equilibrium quantity, payoff and price functions change in the special case of regular network $g$.

**Definition 1.** A regular network is a network where each node has the same number of neighbors. A regular network with nodes of degree $k$ is called a $k$-regular network or regular network of degree $k$.

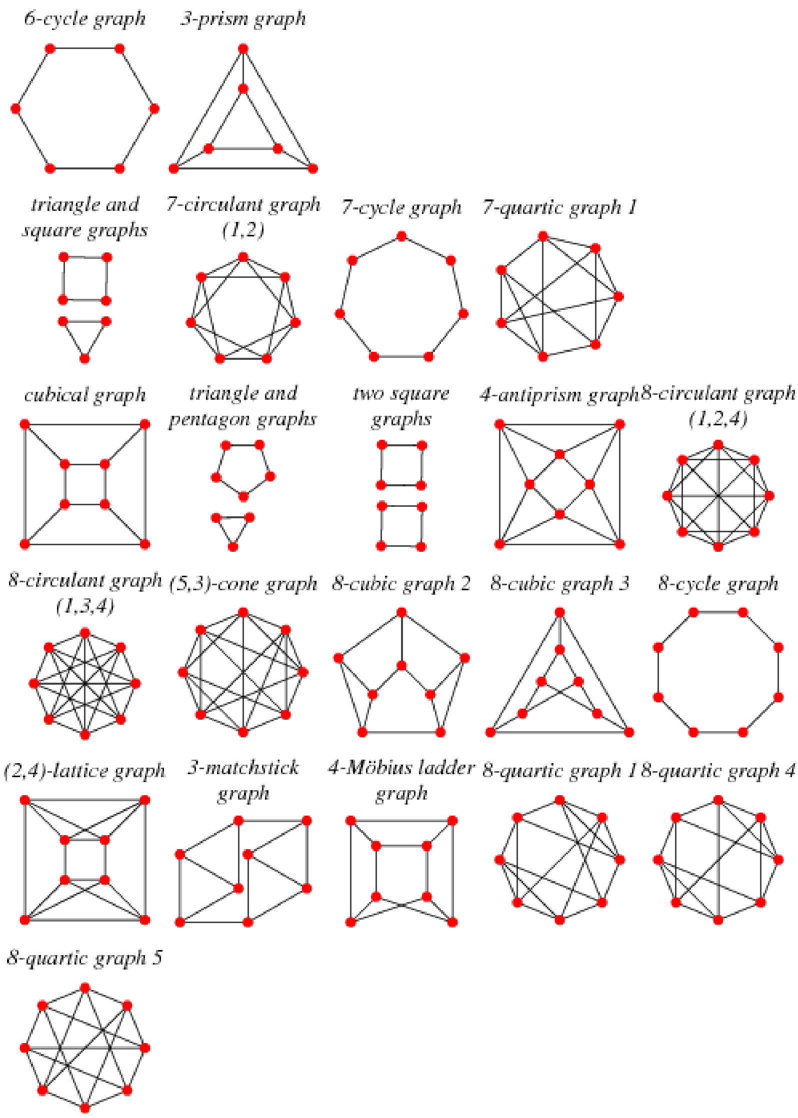


Figure 7: Examples of regular networks.

Write down the optimum quantity (17) for $k$-regular network $g$:

$$\begin{aligned} q_i^*(g) &= \frac{\alpha - \gamma_0 + n\gamma k - \gamma(n-1)k}{n+1} = \\ &= \frac{\alpha - \gamma_0 + \gamma k}{n+1}. \end{aligned} \tag{59}$$

We can notice that in the case of $k$-regular network the equilibrium quantity has simple form. Notice that the equilibrium quantity is linearly dependent upon regularity of the network $k$. And hence the payoff function $\pi_i(g)$ has quadratic dependency upon $k$:

$$\pi_i(g) = q_i^{*2}(g) = \left(\frac{\alpha - \gamma_0}{n+1} + \gamma\frac{k}{n+1}\right)^2. \tag{60}$$

The form of price function for $k$-regular network can be obtained from (5) in the following way:

$$\begin{aligned} p(g) &= \alpha - \sum_{i\in N} q_i^*(g) = \\ &= \alpha - n\frac{\alpha - \gamma_0 + \gamma k}{n+1} = \\ &= \frac{(n+1)\alpha - n(\alpha - \gamma_0 + \gamma k)}{n+1} = \\ &= \frac{\alpha + n\gamma_0 - n\gamma k}{n+1}. \end{aligned} \tag{61}$$

#### 2.5.2 Example

Let us consider the following example. For three cases of network with 11, 12 and 20 players select the parameters according to (23):

$$\alpha = 574,\ \gamma_0 = 21,\ \gamma = 1. \tag{62}$$

Compare equilibrium prices, quantities and profits for different $k$-regular networks with $n$ players. From the Figure 8 below we can observe a positive correlation between degree of node of regular network and the value of the payoff function $\pi_i(g)$. And also we see how fast payoff fails with the increasing number of players $n$ while game parameters do not change. The next Figure 9 demonstrates that the same relations hold for equilibrium quantities. But here we observe that quantity decreases slower than the payoff with the increasing number of players $n$. Figure 10 shows us the connection between price, regularity and the number of players $n$. We can notice that price decreases with the degree of the node in a regular network. And having parameters $\alpha, \gamma_0, \gamma$ fixed price fails with the increasing number of players $n$.

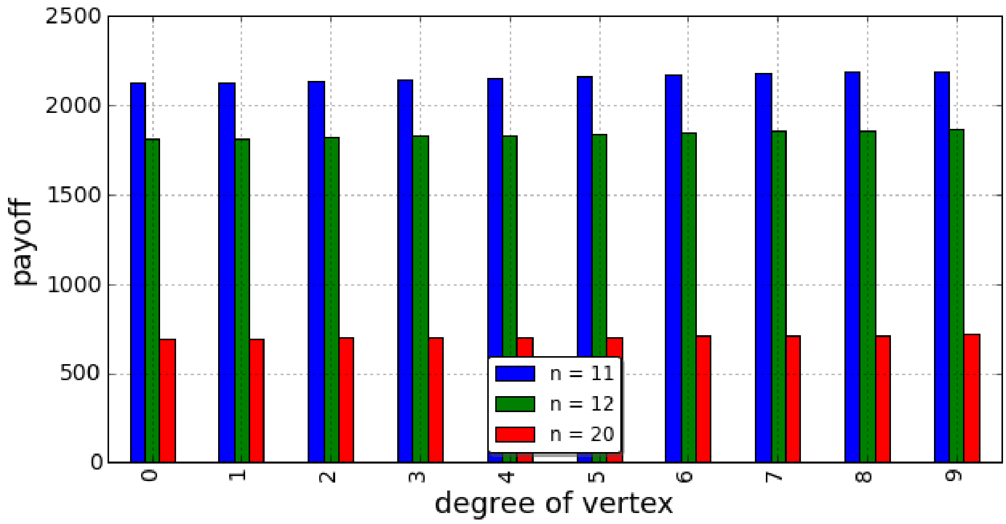


Figure 8: Payoffs for different regular networks.

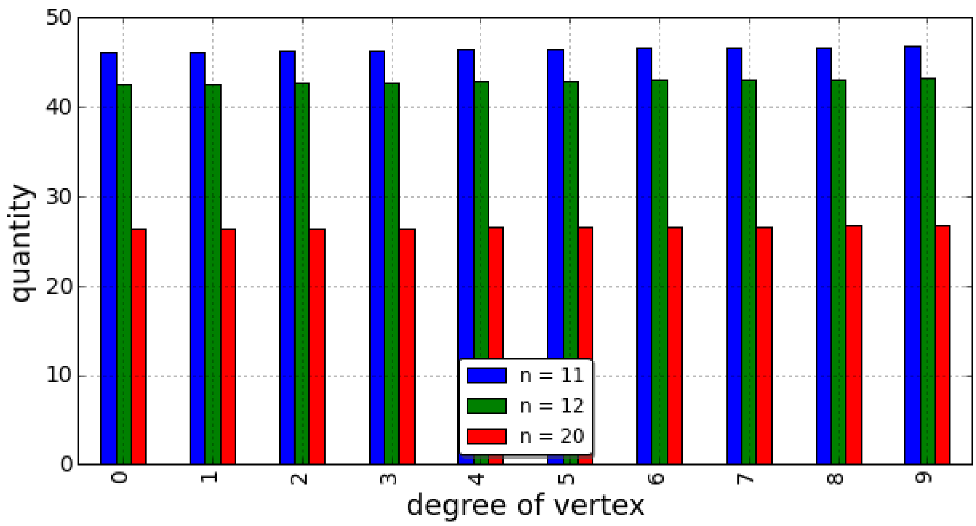


Figure 9: Quantities for different regular networks.

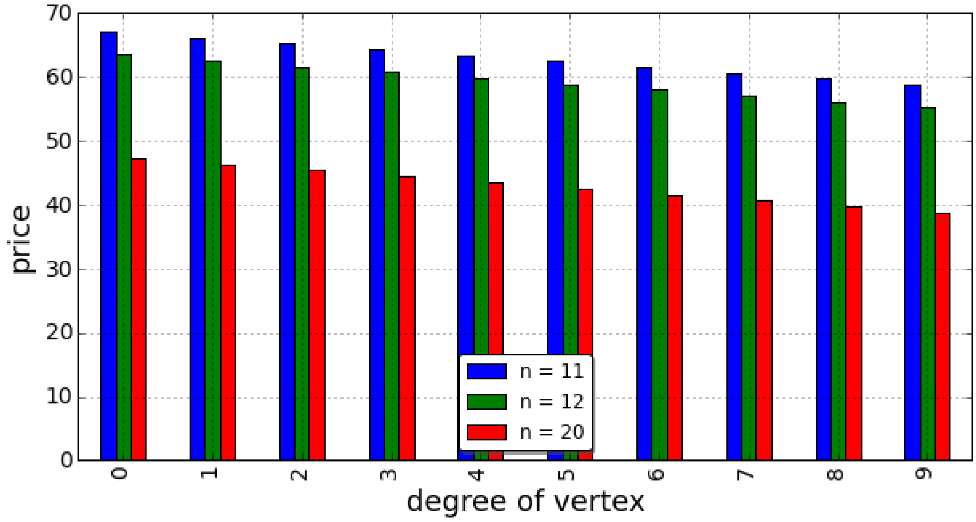


Figure 10: Prices for different regular networks.

## 2.6 Weighted network

The model under consideration ignores distances between firms since it is far from the real life. In this section we cover this issue.

Let us assume that there is given a complete weighted network $g$. Firms are settled in nodes. We may suppose that our network is complete: every firm can reach collaboration with any other firm, but it does not always have a surplus from the collaboration. In order to show this magnitude of profit, each link has a weight which can mean a cost of one supply of resources between nodes which are incident to the link. This cost is the aggregation of length of the link and costs of establishing the link. We do not go further in describing this aggregation because assume that it can be defined differently for each industry.

The first approach is based upon the idea that if two firms are located one close to another they can achieve bigger profit from collaboration with each other neither they are far from each other. We can write this idea in terms of the cost function:

$$c_i(g) = \gamma_0 - \max_{j \in \eta_i(g)} d_{ij}, \tag{63}$$

where $d_{ij}$ is the length of the shortest path from firm $i$ to $j$. Here the word "shortest" should be understood as the way of collaboration of firms $i$ and $j$ which is the most benefit. This "shortest" collaboration can be better than others not only because of the distances of roads on travel map but also because of new conditions in collaboration agreement.

If we repeat the steps which we have done when searched equilibrium at fixed network we will obtain that the equilibrium quantity of firm $i$ for a such cost is the following

$$q_i^*(g) = \frac{\alpha - \gamma_0 + n \max_{j \in \eta_i(g)} d_{ij} - \sum_{k \neq i} \max_{j \in \eta_k(g)} d_{kj}}{n+1}.$$

We can see that the structure of equilibrium quantity does not change after replacing $\eta_j(g)$ with $d_{ij}$. And the player $i$'s payoff stays the same:

$$\pi_i(g) = q_i^{*2}(g).$$

# 3 Cooperative game

When it comes to cooperation the most natural question of players which decided to cooperate in coalition $S \subseteq N$ is how to maximize they common payoff:

$$\pi_S(g, q) = \sum_{i \in S} (p(q) - c_i(g)) q_i. \tag{64}$$

Due to the network structure of our game there are two possible ways to cooperate. Firms can cooperate from the first stage – and play as the one union at network formation step and quantity competition step. We call this type of cooperation a full cooperation. Another way of cooperation is to play individually at the network formation stage and start to cooperate only at quantities competition. We call it quantity cooperation. We consider both types of cooperation. In case of quantity cooperation network is already formed so we need to construct the characteristic function for the game on the fixed network – $v(g, S)$, where $g$ is fixed. In case of full cooperation players are able to choose how to form the network, therefore we need to construct the characteristic function $v(S)$ that depends only upon coalition $S \subseteq N$.

Start from the detail consideration of the quantity cooperation. Cooperative game theory provides a rule of how the total payoff (64) of all players should be split among themselves. It takes into account the relative payoffs that every possible subset of players could get. A cooperative game is defined by the set of players $N$ and the characteristic function $v$, which denotes the power of each coalition $S \subseteq N$. The characteristic function $v$ is a mapping from $2^N$ to real numbers and normalized such that $v(\emptyset) = 0$. To make cooperation interesting for players characteristic function $v$ has to satisfy the superadditivity condition:

$$v(g, S \cup T) \geqslant v(g, S) + v(g, T), \quad \forall S, T \subset N, S \cap T = \emptyset \tag{65}$$

In this section we examine whether the players would want to cooperate and we find allocations of expected sum of players' payoffs (64) using the cooperative game theory tools adapted for network settings.

There are several ways to define a characteristic value of a coalition when we want to consider a cooperative version of a normal-form game. They are:

- The value of the maximin optimization problem, when coalition $S$ attempts to maximize its payoff and complement coalition $N \setminus S$ tries to minimize it.
- The value of the maximization problem, when coalition $S$ maximizes its payoff while players which do not belong to it play fixed strategies, for example Nash equilibrium strategies.
- The equilibrium payoff of coalition $S$ in game in normal form of $|N| - |S| + 1$ players where one player is coalition $S$ and others - are individuals.

We will illustrate the first option in detail. The other two options are discussed further. Since we defined powers of coalitions via characteristic function the next most straightforward question of how payoffs should be divided is that the division should be fair. Such fair allocation is called an imputation.

**Definition 2.** Vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ such that $\sum_{i \in N} \phi_i = v(g, N)$ is called an *imputation* if $\phi_i \geqslant v(g, i), \forall i \in N$ – each player surplus at least what he can get playing individually.

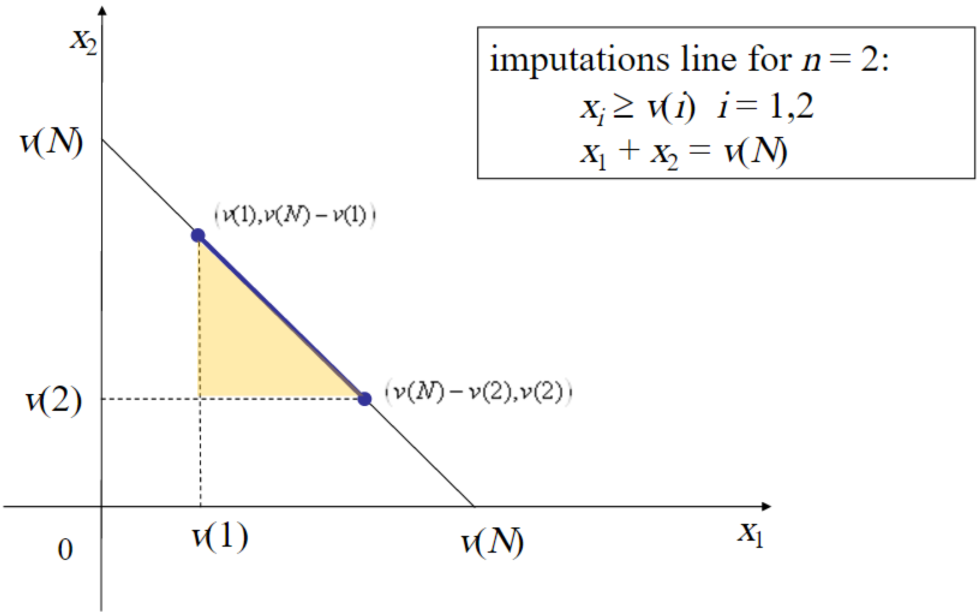


Figure 11: Imputation set for a two-person game.

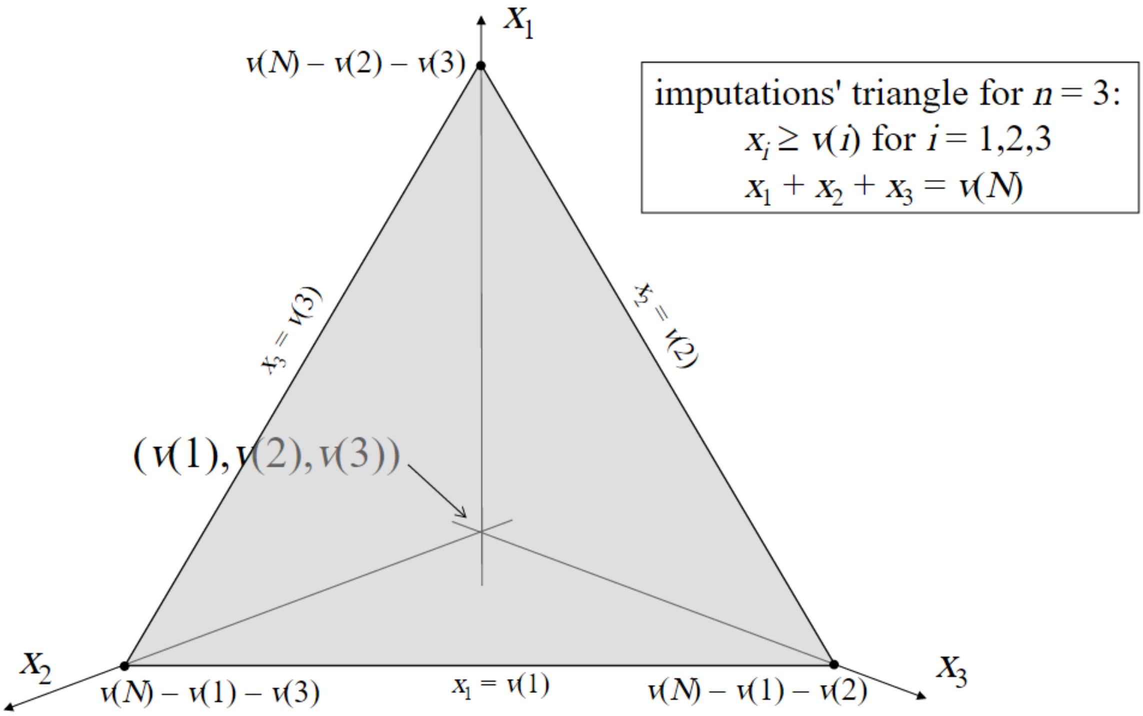


Figure 12: Imputation set for a three-person game.

We will consider two imputations for our game: the Shapley value and the CIS value. Shapley value ensures that players which contribute more to the coalitions than others gain more from cooperation.

**Definition 3.** Given a cooperative game $(N, v)$, the component of the *Shap-*

*ley value* of player $i$ is given by

$$\phi_i(g,v) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N|-|S|-1)! \left[ v(g, S \cup \{i\}) - v(g,S) \right]. \quad (66)$$

**Definition 4.** CIS (center of gravity of the imputation set) for the game $(N, v)$ is defined as

$$CIS_i(g,v) = v(g,\{i\}) + \frac{1}{n} \left[ v(g,N) - \sum_{j \in N} v(g,\{j\}) \right], i \in N. \quad (67)$$

We suppose that when firms cooperate with each other, cost of production of one unit for every firm is the minimal cost in the coalition. Put differently, all firms in the coalition use some resources or technology of the firm with minimal production cost. We can determine a cost function of coalition $S$:

$$c_S(g) = \min_{i \in S} c_i(g) = \gamma - \gamma_0 \max_{i \in S} \eta_i(g). \quad (68)$$

By analogy we determine characteristic function $v(S)$, Shapley and CIS value $\phi_i(v)$.

### 3.1 Maximin

Consider the game of quantities of only two players: coalition $S$ and its complement $N \setminus S$. They play a zero-sum game. It means that the first player $S$ wishes to maximize his payoff and the second player $N \setminus S$ attempts to minimize $S$'s payoff. Since maximin value of a player $S$ is the largest payoff that the player $S$ can be sure to get regardless of the action of the other player $N \setminus S$. Since players in the coalition play as one big firm they need to maximize the sum of their payoffs over the sum of their quantities.

#### 3.1.1 Characteristic function

We start from quantity cooperation. Suppose that network $g$ is fixed. Let us formulate the characteristic function for a fixed network $g$ and coalition $S \subseteq N$:

$$v(g,S) = \max_{q_i,\, i \in S} \min_{q_j,\, j \in N \setminus S} \sum_{i \in S} \pi_i \left(g, q_1, \ldots, q_n\right). \tag{69}$$

Denote the sum of quantities of players in coalition $S$ as $Q_S$, i.e.,

$$\sum_{i \in S} q_i = Q_S, \tag{70}$$

and by analogy

$$\sum_{i \in N \setminus S} q_i = Q_{N \setminus S}.$$

In this notations we may write down the problem (69) in shorter form:

$$v(g,S) = \max_{Q_S} \min_{Q_{N \setminus S}} \sum_{i \in S} \pi_i \left(g, q_1, \ldots, q_n\right) = \max_{Q_S} \min_{Q_{N \setminus S}} Q_S \left(p - c_S(g)\right). \tag{71}$$

Eventually, the characteristic function takes the following form.

$$v(g,S) = \max_{Q_S} \min_{Q_{N \setminus S}} Q_S \left(\alpha - Q_S - Q_{N \setminus S} - c_S(g)\right). \tag{72}$$

Compute the characteristic function for different coalitions. Let us start from grand coalition $S = N$.

$$v(g,N) = \max_{Q_N} Q_N \left(\alpha - Q_N - c_N(g)\right) \tag{73}$$

In this case we have maximization problem of the variable $Q_N$. To find $Q_N$ such that $v(g,N)$ is the maximum, compute the derivative and equal it to zero according to the necessary condition for an extremum:

$$\frac{\partial}{\partial Q_N}\left(Q_N\left(\alpha - Q_N - c_N(g)\right)\right) = 0. \tag{74}$$

We obtain the following:

$$\alpha - c_N(g) - 2Q_N = 0. \tag{75}$$

And hence, the optimum quantity for grand coalition $N$ is

$$Q_N^* = \frac{\alpha - c_N(g)}{2}. \tag{76}$$

Due to the economic approach of our game we should hold the constraint of non-negatively of quantity:

$$Q_N^* \geqslant 0 \Rightarrow \alpha \geqslant c_N(g). \tag{77}$$

Finally, the characteristic function for the grand coalition is the following.

$$v(g, N) = Q_N^{*^2} = \frac{(\alpha - c_N(g))^2}{4} \tag{78}$$

Let us now compute characteristic function for another coalition $S \subset N, S \neq N$. Consider the expression $\big(\alpha - Q_S - Q_{N\setminus S} - c_N(g)\big)$. Notice that while in the game there is only one constraint – non-negativeness of price function, complement coalition $N \setminus S$ is always able to zero maximin value by taking its strategy $Q_{N\setminus S}$ as follows

$$Q_{N\setminus S} = \alpha - c_S(g). \tag{79}$$

Indeed, after substitution the above replacement into (72) we have:

$$v(g,S) = \max_{Q_S} Q_S\,(-Q_S) = 0 \tag{80}$$

We obtain that $v(g,S) = 0, \forall S \subset N, S \neq N$.

Now compute characteristic function for full cooperation in the two-stage game. In this case we need to find the value of the following maximin optimization problem:

$$v(S) = \max_{g_i,\, q_i,\, i \in S} \; \min_{g_j,\, q_j,\, j \in N \setminus S} \sum_{i \in S} \pi_i\,(g, q_1, \ldots, q_n)\,. \tag{81}$$

Suppose that actions $g_S = \{g_i,\, i \in S\}$ and $Q_S$ is already chosen. Then the best option for coalition $N \setminus S$ to choose $g_{N \setminus S}$ is such that nobody from $N \setminus S$ has links with players in coalition $S$. And the action $Q_{N \setminus S}$ for coalition $N \setminus S$ is (79). Consequently we obtain that the characteristic function in the full cooperation coincides with the characteristic function in quantities cooperation.

One more advantage of maximin construction of characteristic function is that we do not need to prove superadditivity of our game $(N, v)$ – it is well known fact. The next aim in cooperative analysis of the game is the Shapley value.

### 3.1.2 Cooperative solution

Let us substitute the value of the characteristic function into (66) to calculate the component of the Shapley value of player $i \in N$:

$$\begin{aligned} &\phi_i(g, N, v) = \\ &= \tfrac{1}{|N|!} \textstyle\sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \left[v(g, S \cup \{i\}) - v(g, S)\right] = \\ &= \tfrac{1}{|N|!} |N \setminus \{i\}|!(|N| - |N \setminus \{i\}| - 1)! \left[v(g, N) - v(g, N \setminus \{i\})\right] = \\ &= \tfrac{1}{|N|!} (|N| - 1)!(|N| - |N| + 1 - 1)! \left[v(g, N) - 0\right] = \\ &= \tfrac{v(g,N)}{|N|} = \tfrac{Q_N^{*2}}{|N|} = \tfrac{(\alpha - c_N(g))^2}{4|N|} = \\ &= \tfrac{(\alpha - \gamma_0 + \gamma \max_{i \in N} \eta_i(g))^2}{4|N|}. \end{aligned} \tag{82}$$

Compute another solution – component of the CIS vector (67) of player $i \in N$:

$$\begin{aligned} CIS_i(g, N, v) &= v(g, \{i\}) + \tfrac{1}{n} \Big[v(g, N) - \textstyle\sum_{j \in N} v(g, \{j\})\Big] = \tfrac{v(g,N)}{|N|} = \\ &= \tfrac{(\alpha - \gamma_0 + \gamma \max_{i \in N} \eta_i(g))^2}{4|N|}. \end{aligned} \tag{83}$$

Notice that the CIS value coincides with the Shapley value. It happens because the value of the characteristic function is not equal to zero only in the case of grand coalition $N$. We may conclude that all players get the same payoff after distribution – no matter whether cost of production is minimal for some player.

## 3.2 Sensitivity analysis

From the expression of the Shapley value (82) we can conclude that all players get equal payoffs. Moreover, these payoffs do not depend on the network topology. On the Figure 13 we can observe two networks – star and complete networks of 10 players. The degree of central player in star network coincides with the degree of every player in complete network. Other players in the star network have less degree but still get the same payoff as the central player.

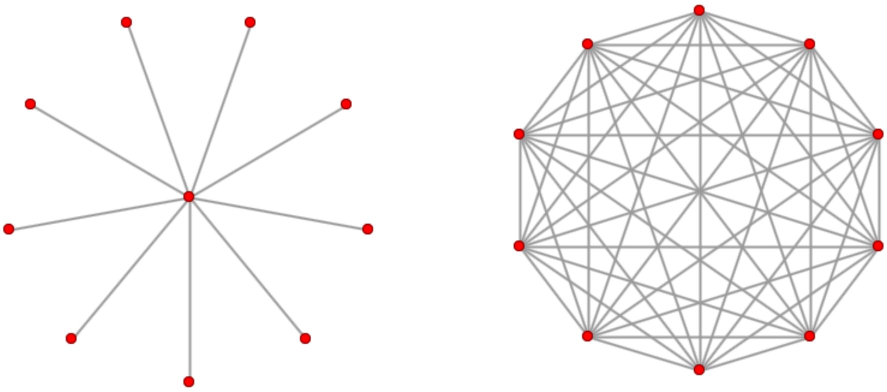

Figure 13: Star and complete networks.

#### 3.2.1 Regular networks

In case of the $k$-regular network the maximum node degree $\max_{i \in N} \eta_i(g)$ coincides with every node degree $\eta_i(g)$, $\forall i \in N$ and equals to $k$ right from the definition of the regular network. Hence obtain the following:

$$\phi_i(g, N, v) = \left( \frac{\alpha - \gamma_0}{n+1} + \gamma \frac{k}{n+1} \right)^2 \quad \forall i \in N.$$

It coincides with the payoff (60) without cooperation at all. Hence, defining the characteristic value as the value minimax optimization problem and by applying the Shapley value (and CIS value) as imputation we obtain that cooperation is not profitable for players relatively to non-coopearation behavior.

# 4 Two-stage oligopoly with offering costs

In previous sections we examined the model of firms competition in one-product market on a network in which link offers were cost-free. Costs were taken only from formed links – when two players offered the collaboration to each other and consequently start of cooperating. But in real life there are situations when the offer of the link could cost something for the firm which proposed it. Even a simple phone call to the potential client may be estimated as an amount of money and represents the offer cost. Another example of the offer cost could arise in situation when firm is trying to figure out how the cooperation could increase its payoff – firms do research of production techniques, legal documents audit and lawyers consultations and so on. It may cost significant and firms should think one more time before deciding to make an offer of collaboration to some other firm.

These thoughts come us to a new cost function of link offer:

$$l_i(g) = \mu \eta_i^{out}(g),\ \mu > 0. \tag{84}$$

Here we use new notation $\eta_i^{out}(g)$ – it is the out-degree of player $i$ in network $g$, the number of collaboration offered by player $i$ indeed. The parameter $\mu$ represents the cost of one offer.

And it leads us to the following payoff function:

$$\pi_i(g, q) = (p(q) - c_i(g))q_i - l_i(g). \tag{85}$$

## 4.1 Equilibrium at fixed network

Here we will make a similar analysis of the non-cooperative game to the analysis provided in the first section but with the new payoff function (85). Our aim here is to study how the new offer's costs will influence payoffs, equilibria and optimal strategies.

We start from finding the optimal quantity on fixed network. To find such optimal quantity, i.e. Nash equilibrium, we should satisfy two conditions: the first order (7) and the second order (8) conditions. But we may notice that if we take the derivative over $q$ of the new payoff function we will obtain exactly the same system of equations as (11). It means that this step has already done and optimal quantity for fixed network in the case of new model coincides with the same one in the old model (20).

### 4.2 Equilibrium in the two-stage game

Repeat the same steps as in the first section. We assume that $g$ is a pairwise network and we will find what conditions on quantity should hold in order to action profile $(g, q)$ be a Nash equilibrium.

Suppose that given a pairwise network $g^*$, deviation of player $i$ from the strategy $g_i^*$ to the strategy $g_i$ expresses in the form (24). Having the same reasoning involving expressions (25), (26), (27), (28). We proceed till the we moment we need to maximize over $g_i$.

Substitute the optimal $q_i$ into the $\pi_i(g^*, q^*||g_i, q_i)$ to obtain $\max_{q_i} \pi_i(g^*, q^*||g_i, q_i)$:

$$\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) =$$
$$= \left[ \alpha - \sum_{j \neq i} q_j^*(g^*) - q_i - \gamma_0 + \gamma \eta_i(g^*||g_i) \right] q_i - l_i(g^*||g_i) = \tag{86}$$

$$= \left[\alpha - \sum_{j\neq i} q_j^*(g^*) - \frac{\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)}{2} - \gamma_0 + \gamma\eta_i(g^*||g_i)\right] \times$$

$$\times \frac{\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)}{2} - \mu\eta_i^{out}(g^*||g_i) =$$

$$= \left(\frac{\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)}{2}\right)^2 - \mu\eta_i^{out}(g^*||g_i) =$$

$$= \frac{1}{4}\left(\alpha - \sum_{j\neq i} q_j^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)\right)^2 - \mu\eta_i^{out}(g^*||g_i) =$$

$$= \frac{1}{4}\left(\alpha - \sum_{j\in N} q_j^*(g^*) + q_i^*(g^*) - \gamma_0 + \gamma\eta_i(g^*||g_i)\right)^2 - \mu\eta_i^{out}(g^*||g_i) =$$

Substitute formula (13) instead of sum $\sum_{j\in N} q_j^*(g^*)$ and formula (20) instead of $q_i^*(g^*)$ into the last expression above:

$$= \frac{1}{4}\Big(\alpha - \frac{n(\alpha-\gamma_0) + \gamma\sum_{i\in N}\eta_i(g^*)}{n+1} +$$

$$+ \frac{\alpha - \gamma_0 + n\gamma\eta_i(g^*) - \gamma\sum_{j\neq i}\eta_j(g^*)}{n+1} - \gamma_0 + \gamma\eta_i(g^*||g_i)\Big)^2 - \mu\eta_i^{out}(g^*||g_i) =$$

$$= \left(\frac{\alpha - \gamma_0 - \gamma\sum_{i\in N}\eta_i(g^*)}{n+1} + \frac{\gamma}{2}\Big(\eta_i(g^*) + \eta_i(g^*||g_i)\Big)\right)^2 - \mu\eta_i^{out}(g^*||g_i) =$$

$$= \left(q_i^*(g^*) + \frac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 - \mu\eta_i^{out}(g^*||g_i)$$

We obtain the maximum over quantity (when player $i$ is deviating in quantity) of the right-hand side of the (28):

$$\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) = \left(q_i^*(g^*) + \frac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 - \mu\eta_i^{out}(g^*||g_i) \tag{87}$$

Now we need to solve the maximization over $g_i$ problem:

$$\max_{g_i}\max_{q_i} \pi_i(g^*, q^*||g_i, q_i) = \tag{88}$$

$$= \max_{g_i}\left[\left(q_i^*(g^*) + \tfrac{\gamma}{2}\Big(\eta_i(g^*||g_i) - \eta_i(g^*)\Big)\right)^2 - \mu\eta_i^{out}(g^*||g_i)\right] \tag{89}$$

Since the inequality below holds

$$\eta_i^{out}(g^*||g_i) \geqslant \eta_i(g^*||g_i)\forall i \in N \;\forall g_i \tag{90}$$

We have the following:

$$\begin{aligned}
&\max_{g_i}\left[\left(q_i^*(g^*)+\tfrac{\gamma}{2}\Big(\eta_i(g^*||g_i)-\eta_i(g^*)\Big)\right)^2-\mu\eta_i^{out}(g^*||g_i)\right]\leqslant\\
&\leqslant\max_{g_i}\left[\left(q_i^*(g^*)+\tfrac{\gamma}{2}\Big(\eta_i(g^*||g_i)-\eta_i(g^*)\Big)\right)^2-\mu\eta_i(g^*||g_i)\right]=\\
&=\max_{g_i}\Big[\,(\eta_i(g^*||g_i))^2+\eta_i(g^*||g_i)\big(\gamma q_i^*(g^*)-\mu-\tfrac{\gamma^2}{2}\eta_i(g^*)\big)+\\
&\qquad+\Big((q_i^*(g^*))^2-\gamma q_i^*(g^*)\eta_i(g^*)-\tfrac{\gamma^2}{4}(\eta_i(g^*))^2\Big)\Big]
\end{aligned}$$

According to the (24) the last expression maximizes when $\eta_i(g^*||g_i) = \eta_i(g^*)$. Finally we obtain the following:

$$\max_{g_i}\max_{q_i}\pi_i(g^*, q^*||g_i, q_i) = (q_i^*(g^*))^2 - \mu\eta_i(g^*) \tag{91}$$

Since

$$\pi_i(g^*, q^*) = (q_i^*(g^*))^2 - \mu\eta_i^{out}(g^*),\ i \in N \tag{92}$$

the Nash equilibrium inequality (25) holds if

$$(q_i^*(g^*))^2 - \mu\eta_i^{out}(g^*) \geqslant (q_i^*(g^*))^2 - \mu\eta_i(g^*),\ i \in N,$$

or

$$\eta_i^{out}(g^*) \leqslant \eta_i(g^*)\, i \in N. \tag{93}$$

But from the definition of $\eta_i^{out}(g)$

$$\eta_i^{out}(g^*) \geqslant \eta_i(g^*)\, i \in N.$$

Consequently, (25) holds only when $\eta_i^{out}(g^*) = \eta_i(g^*)$.

**Proposition 5.** *Pairwise network $g^*$ is the Nash equilibrium if the following condition holds*

$$\eta_i^{out}(g^*) = \eta_i(g^*)\ \forall i \in N. \tag{94}$$

Hence we proved that Nash equilibrium inequality (25) holds for new payoff function. It means that in this new model pairwise networks are the Nash equilibria, i.e. costs of offering the links do not break this equilibria.

### 4.3 Sensitivity analysis

We provide the sensitivity analysis of this new model by analogy with the sensitivity analysis of the previous model in order to compare how the equivalent actions influence quantities, payoffs of different players. As before we will use equilibrium quantities (17) for given network $g$ which are dependent on a network structure so for the simplification we will discard the parameter $q$ in the payoff function (6).

Because the price function in the the model with offering costs depends only on quantities it does not change from the model without offering costs.

Suppose that player $i$ in given network $g$ deletes the link with player $j$, and player $j$ deletes the link with player $i$ as well. After this transformation network $g$ changes and we denote the new network by $\tilde{g}$. An example of such situation is illustrated on Figure 5.

Consider players' payoffs. At first look at the payoff function of player $i$ (19) in the new network $\tilde{g}$. The payoff of player $j$ is changed by the same rule:

$$\begin{aligned}\pi_i(\tilde{g}) &= (q_i^*(\tilde{g}))^2 - \mu\eta_i^{out}(\tilde{g}) = \\ &= (q_i^*(g))^2 - \mu\eta_i^{out}(g) + \gamma\left(1-\frac{2}{n+1}\right)\left(\gamma\left(1-\frac{2}{n+1}\right)-2\right)+\mu = \\ &= \pi_i(g) + \gamma\left(1-\frac{2}{n+1}\right)\left(\gamma\left(1-\frac{2}{n+1}\right)-2\right)+\mu\end{aligned}$$

$$\begin{aligned}\pi_i(\tilde{g}) &= (q_i^*(\tilde{g}))^2 - \mu\eta_i^{out}(\tilde{g}) = \\ &= \left(q_i^*(g) - \gamma\left(1-\frac{2}{n+1}\right)\right)^2 - \mu(\eta_i^{out}(g)-1) = \\ &= \pi_i(g) + \gamma\left(1-\frac{2}{n+1}\right)\left(\gamma\left(1-\frac{2}{n+1}\right)-2q_i^*(g)\right)+\mu\end{aligned}$$

Let us find out under which constraint player $i$ (and player $j$) benefits from the removing the link $(ij)$. The payoff of player $i$ has a negative correlation with the deletion of the collaboration if the next condition holds:

$$\gamma\left(1-\frac{2}{n+1}\right)\left(\gamma\left(1-\frac{2}{n+1}\right)-2q_i^*(g)\right)+\mu < 0. \tag{95}$$

The analysis of the payoff of player $k \neq i,j$ in the network $\tilde{g}$ does not change from the analysis provided for the previous model because $\eta_i^{out}(\tilde{g}) = \eta_i^{out}(g)$. We obtain that if the condition (95) holds, players $i,j$ lose in the network $\tilde{g}$ in comparison to the network $g$. In contrast, player $k$ never loses in the network $\tilde{g}$ in comparison to the network $g$.

## 5 Alternative characteristic functions of cooperative game

As we said in the Section 3 there are several options for defining a characteristic function. Here we construct characteristic function for the two left variants.

### 5.1 Maximization of $S$'s payoff with Nash equilibrium strategies for other individuals

We have a quantity competition game of $|N| - |S| + 1$ players, in which one player is the coalition $S$ and other players choose fixed Nash equilibrium actions (17) as if they played as singletons. Players which do not belong to the coalition $S$ suppose that they play with individuals like they are. It means that they do not know that some players silently formed a coalition and play as one player. For the simplicity let us say that the first $|S|$ players in the initial non-cooperative game of $|N|$ players belong to coalition $S$ in the cooperative game. Then

$$v(g,S) = \max_{q_i, i \in S} \sum_{k \in S} \pi_k(g, q_1, \ldots, q_{|S|}, q^*_{|S|+1}, \ldots, q^*_n). \tag{96}$$

We may rewrite this expression in more convenient form using short notation (70) as follows

$$v(g,S) = \max_{Q_S} \left( \alpha - c_S(g) - Q_S - \sum_{j \in N \setminus S} q^*_j \right) Q_S. \tag{97}$$

According to the necessary condition for an extremum of quadratic function we found optimal quantity of coalition $S$:

$$Q_S^* = \frac{\alpha - c_S(g) - \sum_{j\in N\setminus S} q_j^*}{2}. \tag{98}$$

To express equilibrium quantity $Q_S^*$ through initial parameters of the game, we substitute $q_i^*$ from (17):

$$Q_S^* = \frac{1}{2}\left(\alpha - c_S(g) - \sum_{i\in N\setminus S} \frac{\alpha - \gamma_0 + n\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(g)}{n+1}\right).$$

Let us compute separately the sum in brackets from the last equation:

$$\begin{gathered}
\textstyle\sum_{i\in N\setminus S} \frac{\alpha-\gamma_0+n\gamma\eta_i(g)-\gamma\sum_{j\neq i}\eta_j(g)}{n+1} = \\
= \textstyle\sum_{i\in N\setminus S}\left(\gamma\eta_i(g) + \frac{\alpha-\gamma_0-\gamma\sum_{j\in N}\eta_j(g)}{n+1}\right) = \\
= \gamma\textstyle\sum_{i\in N\setminus S}\eta_i(g) + \frac{1}{n+1}\left(\sum_{i\in N\setminus S}(\alpha-\gamma_0) - \gamma\sum_{i\in N\setminus S}\sum_{j\in N}\eta_j(g)\right) = \\
= \gamma\textstyle\sum_{i\in N\setminus S}\eta_i(g) + \frac{|N|-|S|}{n+1}\left((\alpha-\gamma_0) - \gamma\sum_{j\in N}\eta_j(g)\right) = \\
= \frac{n-|S|}{n+1}\left((\alpha-\gamma_0) - \gamma\textstyle\sum_{j\in S}\eta_j(g)\right) + \frac{|S|+1}{n+1}\gamma\textstyle\sum_{i\in N\setminus S}\eta_i(g)
\end{gathered}$$

Now we are able to substitute this sum into (98).

$$\begin{gathered}
Q_S^* = \tfrac{1}{2}\left(\alpha - c_S(g) - \tfrac{n-|S|}{n+1}\left((\alpha-\gamma_0)-\gamma\textstyle\sum_{j\in S}\eta_j(g)\right) - \tfrac{|S|+1}{n+1}\gamma\textstyle\sum_{i\in N\setminus S}\eta_i(g)\right) = \\
= \tfrac{1}{2}\left(\alpha - \gamma_0 + \gamma\max_{j\in S}\eta_j(g) - \tfrac{n-|S|}{n+1}\left((\alpha-\gamma_0)-\gamma\textstyle\sum_{j\in S}\eta_j(g)\right) - \tfrac{|S|+1}{n+1}\gamma\textstyle\sum_{i\in N\setminus S}\eta_i(g)\right) = \\
= \tfrac{1}{2}\left(\tfrac{|S|+1}{n+1}(\alpha-\gamma_0) + \gamma\max_{j\in S}\eta_j(g) - \tfrac{n-|S|}{n+1}\gamma\textstyle\sum_{j\in S}\eta_j(g) - \tfrac{|S|+1}{n+1}\gamma\textstyle\sum_{i\in N\setminus S}\eta_i(g)\right)
\end{gathered}$$

For grand coalition $N$, the sum of $q_i^*$ there is no in the formula (97). So the expression for optimal $Q_N^*$ is significantly shorter and coincides with corresponding expression for the first case of constructing characteristic function – (76) and has the following form:

$$Q_N^* = \frac{\alpha - c_N(g)}{2}. \tag{99}$$

## 5.2 Equilibrium in the game with $|N| - |S| + 1$ players

Here to determine characteristic function $v(g, S)$ we consider the game of $(|N| - |S| + 1)$ players, where one player represents coalition $S$ and other players are individuals. Unlike from the previous way of defining characteristic function here all players know that one player actually represents a group of players. Suppose that first $|S|$ players of $N$ are in coalition $S$. Players have the following payoff functions:

$$\pi_S(g, g, Q_S, q_{|S|+1}, \ldots, q_n) = (p - c_S(g))\, Q_S, \text{ for player-coalition } S$$

$$\pi_i(g, g, q_{|S|+1}, \ldots, q_n) = (p - c_i(g))\, q_i, \forall i \in N \setminus S.$$

Then the characteristic function $v(g, S)$ is the equilibrium payoff of coalition $S$ when quantities of production are fixed optimal quantities in Nash equilibrium.

$$v(g, S) = \left( \alpha - Q_S^* - \sum_{j \in N \setminus S} q_j^* - c_S(g) \right) Q_S^* \tag{100}$$

Now let us find Nash equilibrium in this game:

$$\begin{cases} \frac{\partial \pi_S(g}{\partial Q_S} = \alpha - 2Q_S - \sum_{j \in N \setminus S} q_j - c_S(g) = 0, & \text{for player } S \\ \frac{\partial \pi_i(g)}{\partial q_i} = \alpha - Q_S - q_i - \sum_{j \in N \setminus S} q_j - c_i(g) = 0, & \forall i \in N \setminus S \end{cases}$$

$$\begin{cases} \alpha - Q_S - \sum_{j \in N} q_j - c_S(g) = 0 \\ \alpha - q_i - \sum_{j \in N} q_j - c_i(g) = 0, \quad \forall i \in N \setminus S \end{cases} \tag{101}$$

Here we can notice an interesting connection between individual quantity $q_i$, $\forall i \in N \setminus S$ and coalition quantity $Q_S$ from the system (101) above

$$\alpha - c_S(g) - Q_S = \sum_{j \in N} q_j = \alpha - c_i(g) - q_i.$$

From which we obtain the following:

$$c_S(g) + Q_S = c_i(g) + q_i, \ \forall i \in N \setminus S.$$

When we sum equations for $i \in N \setminus S$ we obtain the following.

$$(n - |S|)\alpha - \sum_{i \in N \setminus S} q_i - (n - |S|) \sum_{j \in N} q_j - \sum_{i \in N \setminus S} c_i(g) = 0$$

After addition first equation to this sum we get this expression.

$$(n - |S| + 1)\alpha - (n - |S| + 2) \sum_{j \in N} q_j - c_S(g) - \sum_{i \in N \setminus S} c_i(g) = 0$$

From this equation we express the sum of all quantities.

$$\sum_{j \in N} q_j = \frac{1}{n - |S| + 2} \left( (n - |S| + 1)\alpha - c_S(g) - \sum_{i \in N \setminus S} c_i(g) \right)$$

Finally, let us substitute the last expression into the first equation of the system.

$$Q_S = \alpha - \sum_{j \in N} q_j - c_S(g)$$

$$Q_S = \alpha - c_S(g) - \frac{(n - |S| + 1)\alpha - c_S(g) - \sum_{i \in N \setminus S} c_i(g)}{n - |S| + 2}$$

We found optimal $Q_S^*$ in Nash equilibrium.

$$Q_S^* = \frac{\alpha - (n - |S| + 1)c_S(g) + \sum_{i \in N \setminus S} c_i(g)}{n - |S| + 2}$$

For the case of grand coalition $N$ we have the following short form, which also coincides with corresponding expressions for the other ways of constructing characteristic function.

$$Q_N^* = \frac{\alpha - c_N(g)}{2}$$

By substituting the expression of sum of quantities into the remaining equations of the system.

$$q_i = \alpha - \sum_{j \in N} q_j - c_i(g), \ \forall i \in N \setminus S$$

$$q_i = \alpha - c_i(g) - \frac{(n - |S| + 1)\alpha - c_S(g) - \sum_{j \in N \setminus S} c_j(g)}{n - |S| + 2}, \ \forall i \in N \setminus S$$

Finally we found optimal quantities for remaining individual players $i \in N \setminus S$.

$$q*_i = \frac{\alpha + c_S(g) - (n - |S| + 1)c_i(g) + \sum_{j \in N \setminus (S \cup \{i\})} c_j(g)}{n - |S| + 2}, \ \forall i \in N \setminus S$$

Now we are able to write down characteristic function because we already know all optimal quantities.

$$v(g, S) = \left( \alpha - Q_S^* - \sum_{j \in N \setminus S} q_j^* - c_S(g) \right) Q_S^* = Q_S^{*^2} \tag{102}$$

# 6 Conclusion

In this thesis we investigated the one-product market competition in quantities of $n$ firms. In our model firms are able to establish collaborations between themselves and chose quantities of production. The set of pairwise collaborations defines the network. As the payoff function we use profit from production and selling goods. The network effect appears in payoff function and more precisely in marginal costs. The marginal cost of firm is negatively correlated with the number of formed collaboration links of the firm. We considered both non-cooperative and cooperative games and used Nash equilibrium as a non-cooperative solution of the two-stage game and the Shapley value as a cooperative solution. At first we found the equilibrium quantity of player when the network is fixed. Then we considered the two-stage game and found the equilibrium strategies: they are pairwise networks and equilibrium quantities which coincides with the equilibrium quantities for fixed network. Since there are too many equilibria in the two-stage game we provided an analysis of some specific networks and compared different configurations. We characterized and compared firms' payoffs under different collaboration structures: the empty network, a regular network, the complete network, and a star-like network. To uncover how the collaborations influence the price function and payoffs of players we provided the sensitivity analysis of removing and adding a collaboration link. There was found the amount of surplus for players who were involved in the establishing of the new collaboration link. And it was found how much the common market price decreases with the degree of the node in a regular network and obtained that price increases with the number of firms in the market. For the special case of the regular networks we found explicit formulas of the equilibrium quantity and the price and provided a sensitivity analysis, in which we showed on the numerical example how the degree of node and the number of players affect on the market price, quantities and payoffs of players. We also introduced an approach of the model to the weighted networks and showed that in this case the structure of the equilibrium quantity was not changed.

After the non-cooperative game we considered a cooperative game. We illustrated options of choosing the characteristic function. We defined the value of the characteristic function as the solution of the maximin optimization problem. Then we found the Shapley value and the CIS value as solutions. It did happen that they coincide. It means that in the cooperative game all players get equal payoffs. Moreover, for the regular network the payoff of the player in the cooperative game coincides with his payoff in the non-cooperative game. We also obtained that in the cooperative game for the maximin characteristic function, players are indifferent to the network structure whether maximum degree of the node in the network does not change.

Next we additionally examined an extended version of the model: two-stage oligopoly with offering costs. This model can emerge from numerous economic applications when the offer of the collaboration leads to extra costs, without confidence that it will be accepted. We justified that the equilibrium strategies for the prior model are the equilibrium strategies but with one condition.

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